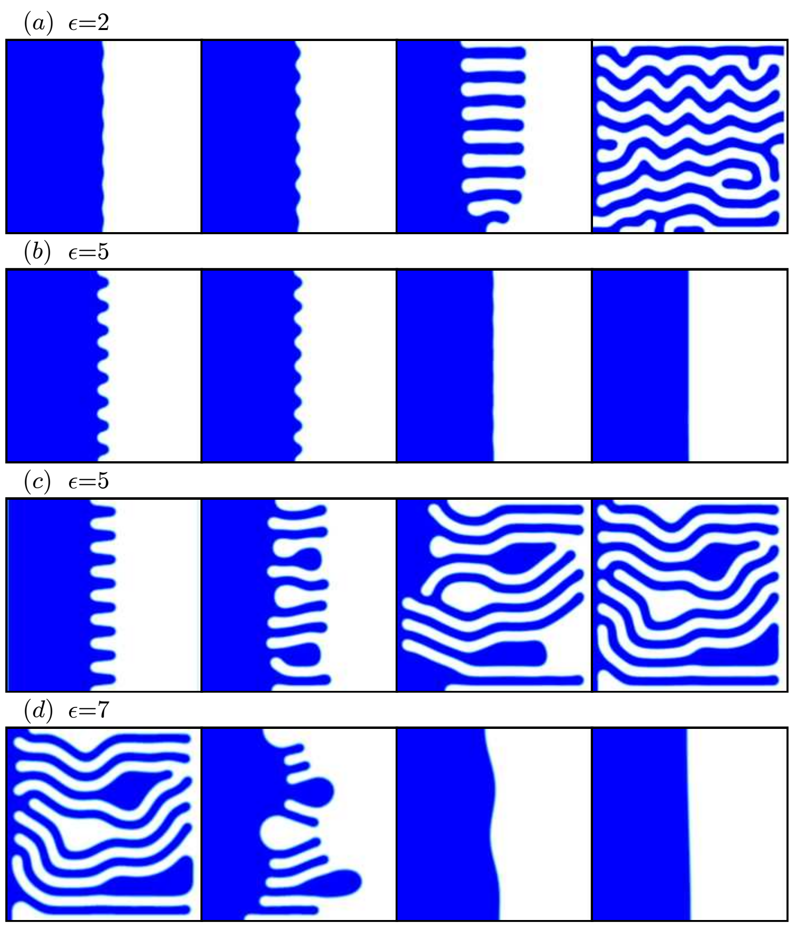
Linear and Nonlinear Front Instabilities in Bistable Systems

Aric Hagberg, hagberg@lanl.gov

Pattern formation phenomena in systems with two stable states (bistable) are determined to a large extent by instabilities of the fronts between the two states. Fronts between stable uniform states can go through transverse instabilities leading to stationary labyrinthine patterns, or through non-equilibrium Ising-Bloch (NIB) bifurcations resulting in traveling wave phenomena such as spiral waves.

We are studying the stability of planar fronts to transverse perturbations using the Swift-Hohenberg equation and a model for urban population spread. Contiguous to the linear transverse instability that has been studied in earlier works, a parameter range is found where planar fronts are *linearly stable* but *nonlinearly unstable*: transverse perturbations beyond some critical size grow rather than decay.

The nonlinear front instability is a result of the coexistence of stable planar fronts and stable large-amplitude patterns. While the linear transverse instability leads to labyrinthine patterns through fingering and tip splitting, the nonlinear instability often evolves to spatial mixtures of stripe patterns and irregular regions of the uniform states.



Numerical solutions of the Swift-Hohenberg equation demonstrating the linear and nonlinear front instabilities. The single control parameter is the linear driving coefficient ϵ. (a) In the parameter region of linear front instability small perturbations on a front grow and form a labyrinthine pattern. In the parameter region of nonlinear front instability, (b) small perturbations do not grow, but (c) large perturbations are sufficient to create a patterned state. (d) In the stable parameter range fronts are globally stable; An initial pattern state returns to a front. Time increases in the frames from left to right.

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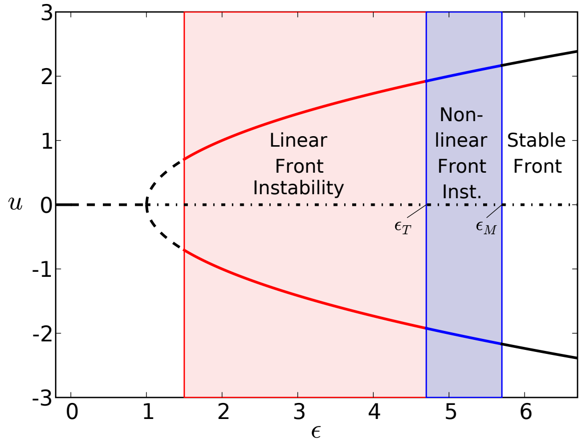

Bifurcation diagram for uniform solutions of the Swift-Hohenberg equation with regions of linear and nonlinear front instabilities. The familiar linear transverse front instability is only found for a range of the control parameter up to ϵ_T. Beyond that range, for $\epsilon \in [\epsilon_T, \epsilon_M]$ fronts are linearly stable but nonlinearly unstable; small perturbations shrink but sufficiently large perturbations grow.

The Swift-Hohenberg equation is

$$u_t = \epsilon u - (\nabla^2 + 1)^2 u - u^3,$$

where u is a real scalar field and ϵ is the bifurcation control parameter. The zero solution $u = 0$ loses stability to finite-wavenumber perturbations at $\epsilon = 0$, and goes through a pitchfork bifurcation at $\epsilon = 1$. The two uniform states, $u_\pm = \pm\sqrt{\epsilon - 1}$, that appear above $\epsilon = 1$ are unstable to finite-wavenumber perturbations but become stable above $\epsilon = 3/2$.

The front solutions are linearly unstable to transverse perturbations up to a threshold $\epsilon = \epsilon_T$ for which we have an analytic formula. This linear instability is demonstrated in Figure (a) (on opposite page). Beyond ϵ_T, the linear transverse instability disappears; small transverse perturbations of the front decay out as Figure (b) shows. The front, however, remains unstable to finite-size perturbations, implying a *nonlinear* transverse instability. The instability is demonstrated in Figure (c) which also shows the asymptotic pattern that develops - a spatial mixture of parallel stripes and regions of the two stable uniform states. The nonlinear transverse instability disappears at a yet higher threshold, ϵ_M, which we calculate numerically using a Lyapunov functional. Figure (d) demonstrates the global front stability above ϵ_M by showing the retraction of a pattern state to a planar front.

The asymptotic patterns that develop from nonlinear transverse front instabilities can differ considerably from the labyrinthine patterns that develop from linear front instabilities. The linear stability of the fronts along with the stability of the symmetric uniform states often favor the formation of uniform solution regions intermingled with stripes (see Figure (c) on the opposite page).

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Co-investigators:

L. Pismen Department of Chemical Engineering and Minerva Center for Nonlinear Physics of Complex Systems, Technion, Haifa, 32000, Israel

A. Yochelis Department of Physics, University of California at Berkeley, Berkeley, CA 94720

C. Elphick, Centro de Fisica No Lineal y Sistemas Complejos de Santiago, Casilla 17122, Santiago, Chile

E. Meron and H. Yizhaq, J. Blaustein Institute for Desert Research, Ben-Gurion University, Sede Boker Campus 84990

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References

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